Exercise 16

In a murder investigation, the temperature of the corpse was 32.5°C at 1:30 pm and 30.3°C an hour later. Normal body temperature is 37.0°C and the temperature of the surroundings was 20.0°C. When did the murder take place?

Solution

Assume that the rate of decrease of the corpse's temperature is proportional to the difference between the corpse's temperature and the surrounding temperature T_s.

$$\frac{dT}{dt} \propto -(T - T_s)$$

The minus sign is included so that when the surroundings are cooler (hotter) than the corpse, dT/dt is negative (positive). Change this proportionality to an equation by introducing a positive constant k.

$$\frac{dT}{dt} = -k(T - T_s)$$

To solve this differential equation for T, make the substitution $y = T - T_s$.

$$\frac{dT}{dt} = -ky$$

Differentiate both sides of the substitution with respect to t to write the derivative in terms of y: $\frac{dy}{dt} = \frac{d}{dt}(T - T_s) = \frac{dT}{dt}$.

$$\frac{dy}{dt} = -ky$$

Divide both sides by y.

$$\frac{1}{y}\frac{dy}{dt} = -k$$

Rewrite the left side by using the chain rule.

$$\frac{d}{dt}\ln y = -k$$

The function you take a derivative of to get $-k$ is $-kt + C$, where C is any constant.

$$\ln y = -kt + C$$

Exponentiate both sides to get y.

$$e^{\ln y} = e^{-kt+C}$$

$$y = e^C e^{-kt}$$

Use a new constant A for e^C.

$$y(t) = Ae^{-kt}$$

Now that the differential equation has been solved, change back to the original variable T, the corpse's temperature.

$$T - T_s = Ae^{-kt}$$

As a result,

$$T(t) = T_s + Ae^{-kt}.$$

Since the surrounding temperature is 20.0°C, $T_s = 20$.

$$T(t) = 20 + Ae^{-kt}$$

Let $t = 0$ be the time of murder. The corpse's temperature at this time is 37°C.

$$37 = 20 + Ae^{-k(0)} \quad \rightarrow \quad A = 37 - 20 = 17$$

Consequently,

$$T(t) = 20 + 17e^{-kt}.$$

Let t_0 be the time in hours from the murder until 1:30 PM. Use the two given temperatures at the two given times to form a system of equations for the unknowns, k and t_0.

$$\left.\begin{aligned} T(t_0) = 20 + 17e^{-kt_0} &= 32.5 \\ T(t_0 + 1) = 20 + 17e^{-k(t_0+1)} &= 30.3 \end{aligned}\right\}$$

$$\left.\begin{aligned} 17e^{-kt_0} &= 12.5 \\ 17e^{-k(t_0+1)} &= 10.3 \end{aligned}\right\}$$

$$\left.\begin{aligned} 17e^{-kt_0} &= 12.5 \\ 17e^{-kt_0}e^{-k} &= 10.3 \end{aligned}\right\}$$

Substitute the first equation into the second one.

$$(12.5)e^{-k} = 10.3$$

$$e^{-k} = \frac{10.3}{12.5}$$

$$e^{-k} = 0.824$$

$$\ln e^{-k} = \ln 0.824$$

$$(-k)\ln e = \ln 0.824$$

$$k = -\ln 0.824$$

Substitute this result back into the first equation.

$$17e^{-kt_0} = 12.5$$

$$17e^{-(-\ln 0.824)t_0} = 12.5$$

$$17e^{\ln(0.824)^{t_0}} = 12.5$$

$$17(0.824)^{t_0} = 12.5$$

$$0.824^{t_0} = \frac{12.5}{17}$$

$$0.824^{t_0} = \frac{25}{34}$$

$$\ln 0.824^{t_0} = \ln \frac{25}{34}$$

$$t_0 \ln 0.824 = \ln \frac{25}{34}$$

$$t_0 = \frac{\ln \frac{25}{34}}{\ln 0.824} \approx 1.58837$$

Find out how many minutes 0.58837 hours is.

$$0.58837 \text{ hours} \times \frac{60 \text{ minutes}}{1 \text{ hour}} \approx 35.3024 \text{ minutes}$$

Find out how many seconds 0.3024 minutes is.

$$0.3024 \text{ minutes} \times \frac{60 \text{ seconds}}{1 \text{ minute}} \approx 18.1411 \text{ seconds}$$

An hour and 35 minutes and 18 seconds prior to 1:30 PM is 11:54:42 AM.